

Analytics of the Melitz model

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1 Assumptions

- 2 symmetric countries
- CES utility function:

$$U = \left[\int_{\omega \in \Omega} q(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{\sigma}{\sigma-1}}$$

with σ the elasticity of substitution between varieties and Ω the (endogenous) mass of available goods

- R the nominal income, equal in equilibrium to wL (equal to L when w is normalized to unity)
- A large unbounded pool of prospective entrants
- Sunk cost to produce f_e (in labor units)
- Once the sunk cost paid, productivity φ is the realization of a random variable. Distribution $g(\varphi)$ with positive support $(0, \infty)$ and a continuous cumulative distribution $G(\varphi)$
- Constant death probability δ but not discounting factor
- Fixed cost to produce f and fixed cost to export f_{ex} (in labor units)
- Per-unit iceberg trade cost $\tau > 1$

2 Results

2.1 Demand functions

Write the Lagrangien:

$$L = \left[\int_{\omega \in \Omega} q(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{\sigma}{\sigma-1}} - \lambda \left[\int_{\omega \in \Omega} p(\omega)q(\omega) - R \right]$$

First order conditions:

$$\begin{aligned} \frac{\partial L}{\partial q(\omega)} &= q(\omega)^{\frac{-1}{\sigma}} \left[\int_{\omega \in \Omega} \omega^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{1}{\sigma-1}} - \lambda p(\omega) = 0 \\ \Leftrightarrow q(\omega)^{\frac{-1}{\sigma}} U^{\frac{1}{\sigma}} &= \lambda p(\omega) \\ \Leftrightarrow p(\omega)q(\omega) &= U \lambda^{-\sigma} p(\omega)^{1-\sigma} \end{aligned}$$

Integrate over Ω :

$$\int_{\omega \in \Omega} p(\omega)q(\omega) d\omega = U \lambda^{-\sigma} \int_{\omega \in \Omega} p(\omega)^{1-\sigma} d\omega$$

and

$$U = \left[\int_{\omega \in \Omega} q(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{\sigma}{\sigma-1}} = U \lambda^{-\sigma} \left[\int_{\omega \in \Omega} p(\omega)^{1-\sigma} d\omega \right]^{\frac{\sigma}{\sigma-1}}$$

Remember that $R \equiv PC$, this gives:

$$P = \left[\int_{\omega \in \Omega} p(\omega)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}}$$

From: $R = U \lambda^{-\sigma} P^{1-\sigma}$ and $q(\omega)^{\frac{-1}{\sigma}} U^{\frac{1}{\sigma}} = \lambda p(\omega)$, one obtains the demand function:

$$q(\omega) = \left(\frac{p(\omega)}{P} \right)^{-\sigma} \frac{R}{P}$$

\Rightarrow Everything else being equal, a 1% rise in $p(\omega)$ reduces demand $q(\omega)$ by $\sigma\%$ (ie σ measures the price-elasticity of demand)

\Rightarrow The demand $q(\omega)$ also depends on the consumer's purchasing power R/P

$\Rightarrow \frac{q(\omega)}{q(\omega')} = \left(\frac{p(\omega)}{p(\omega')} \right)^{-\sigma}$: Increasing the relative price of the ω variety by 1% reduces the

relative demand for this variety by $\sigma\%$ (elasticity of substitution)

2.2 Optimal price

2.2.1 Domestic firms

Start from the firm's profit function:

$$\pi(\omega) = p(\omega)q(\omega) - w \left(f + \frac{q(\omega)}{\varphi(\omega)} \right)$$

Maximize with respect to price given demand function: $q(\omega) = \left(\frac{p(\omega)}{P} \right)^{-\sigma} \frac{R}{P}$

Remember that monopolistic competition implies that the firm considers aggregate prices as given (check what happens if the firm is not atomistic)

First order condition:

$$\frac{\partial \pi(\omega)}{\partial p(\omega)} = P^{\sigma-1} R \left[(1 - \sigma)p(\omega)^{-\sigma} + \frac{w}{\varphi(\omega)} \sigma p(\omega)^{-\sigma-1} \right] = 0$$

Or after rearranging:

$$p(\omega) = \underbrace{\frac{\sigma}{\sigma - 1}}_{\text{Mark-up}} \underbrace{\frac{w}{\varphi(\omega)}}_{\text{Marginal cost}}$$

\Rightarrow Constant mark-up, decreasing in σ (more differentiated goods provide the firm with a larger market power (lower price elasticity of demand), which allows her to set higher prices

\Rightarrow Marginal cost is decreasing in the firm's individual productivity

2.2.2 Exporting firms

For exporting firms, there are two decision variables (domestic price $p_d(\omega)$ and export price $p_x(\omega)$) and the profit function rewrites:

$$\pi(\omega) = p_d(\omega)q_d(\omega) + p_x(\omega)q_x(\omega) - w \left(f + f_{ex} + \frac{q_d(\omega)}{\varphi(\omega)} + \tau \frac{q_x(\omega)}{\varphi(\omega)} \right)$$

with:

$$q_d(\omega) = \left(\frac{p_d(\omega)}{P} \right)^{-\sigma} \frac{R}{P} \quad q_x(\omega) = \left(\frac{p_x(\omega)}{P^*} \right)^{-\sigma} \frac{R}{P^*}$$

where R^* and P^* denote the revenue and price level in the export market (equal to R and P , respectively, with symmetric countries).

Since the profit function is separable, optimization gives similar results as in the case of purely domestic firms:

$$p_d(\omega) = \frac{\sigma}{\sigma - 1} \frac{w}{\varphi(\omega)} \quad p_x(\omega) = \frac{\sigma}{\sigma - 1} \tau \frac{w}{\varphi(\omega)}$$

The (FOB) export price is equal to the domestic price since the elasticities of substitution are assumed identical in both countries. The CIF export price is equal to the FOB price times the per-unit trade cost. Everything else equal, this means that a foreign firm tends to be less competitive than an equally-productive domestic firm, that does not face the transport cost.

2.3 Aggregation in closed economy

The aggregate price is given by:

$$\begin{aligned} P &= \left[\int_{\omega \in \Omega} p(\omega)^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \\ &= \left[\int_0^\infty p(\varphi)^{1-\sigma} M \mu(\varphi) d\varphi \right]^{\frac{1}{1-\sigma}} \\ &= \left[\int_0^\infty \left(\frac{\sigma}{\sigma - 1} \frac{w}{\varphi} \right)^{1-\sigma} M \mu(\varphi) d\varphi \right]^{\frac{1}{1-\sigma}} \\ &= M^{\frac{1}{1-\sigma}} p(\tilde{\varphi}) \end{aligned}$$

where M is the mass of firms/goods available in the market, $\mu(\varphi)$ is the distribution of productivities, and

$$\tilde{\varphi} = \left[\int_0^\infty \varphi^{\sigma-1} \mu(\varphi) d\varphi \right]^{\frac{1}{\sigma-1}}$$

is a weighted average of the firm productivity levels.

Together with M , $\tilde{\varphi}$ entirely determines aggregate variables:

$$\begin{aligned}
R &= \int_{\omega \in \Omega} p(\omega)q(\omega)d\omega \\
&= \int_0^\infty p(\varphi)q(\varphi)M\mu(\varphi)d\varphi \\
&= R \left(\frac{\sigma - 1}{\sigma} P \right)^{\sigma-1} M\tilde{\varphi}^{\sigma-1} \\
&= Mp(\tilde{\varphi})q(\tilde{\varphi}) \\
Q &= \frac{R}{P} \\
&= M^{\frac{\sigma}{\sigma-1}}q(\tilde{\varphi}) \\
\Pi &= \int_{\omega \in \Omega} \pi(\omega)d\omega \\
&= \int_0^\infty \pi(\varphi)M\mu(\varphi)d\varphi \\
&= \int_0^\infty \left(\frac{p(\varphi)q(\varphi)}{\sigma} - f \right) M\mu(\varphi)d\varphi \\
&= M\pi(\tilde{\varphi})
\end{aligned}$$

2.4 Zero Cutoff Profit condition

Firms remain on the market if and only if the profit entailed from producing in a given period is positive: $\pi(\varphi) \geq 0$. Since profits are increasing in productivity, any firm with a productivity level higher than $\varphi^* = \inf\{\varphi : \pi(\varphi) \geq 0\}$ stays in the market after drawing her productivity. This implies the following “Zero Cutoff Profit” condition:

$$\pi(\varphi^*) = 0$$

The aggregate productivity level rewrites:

$$\tilde{\varphi}(\varphi^*) = \left[\frac{1}{1 - G(\varphi^*)} \int_{\varphi^*}^\infty \varphi^{\sigma-1} g(\varphi) d\varphi \right]^{\frac{1}{\sigma-1}}$$

and the average profit and revenue levels:

$$\begin{aligned}\bar{r} &= r(\tilde{\varphi}) = \left[\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*} \right]^{\sigma-1} r(\varphi^*) \\ \bar{\pi} &= \pi(\tilde{\varphi}) = \left[\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*} \right]^{\sigma-1} \frac{r(\varphi^*)}{\sigma} - f\end{aligned}$$

The ZCP condition thus becomes:

$$\pi(\varphi^*) = 0 \Leftrightarrow r(\varphi^*) = \sigma f \Leftrightarrow \bar{\pi} = f \left[\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*} \right)^{\sigma-1} - 1 \right]$$

2.5 Free Entry condition

For entry to be profitable, expected future profits must be positive.

Let's define the present value of the average profit flows:

$$\begin{aligned}\bar{v} &= \sum_{t=0}^{\infty} (1-\delta)^t \bar{\pi} = \frac{1}{\delta} \bar{\pi} \\ \bar{v} &= \int_{\varphi^*}^{\infty} v(\varphi) \mu(\varphi) d\varphi\end{aligned}$$

The free entry condition implies:

$$p_{in} \bar{v} = f_e \Leftrightarrow \bar{\pi} = \frac{\delta f_e}{1 - G(\varphi^*)}$$

2.6 Equilibrium in closed economy

The cutoff productivity, together with average profits, is determined by the system formed by the ZCP and FE conditions:

$$\begin{aligned}(ZE) &= \bar{\pi} = \frac{\delta f_e}{1 - G(\varphi^*)} \\ (ZCP) &= \bar{\pi} = f \left[\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*} \right)^{\sigma-1} - 1 \right]\end{aligned}$$

The equilibrium cutoff level is unique iff $j(\varphi^*) \equiv [1 - G(\varphi^*)] \left[\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*} \right)^{\sigma-1} - 1 \right]$ is monotonically decreasing from infinity to zero on $(0, \infty)$:

$$\begin{aligned} j'(\varphi^*) &= -g(\varphi^*) \left[\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*} \right)^{\sigma-1} - 1 \right] + [1 - G(\varphi^*)](\sigma - 1) \left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*} \right)^{\sigma-2} \frac{\tilde{\varphi}'(\varphi^*)\varphi^* - \tilde{\varphi}(\varphi^*)}{\varphi^{*2}} \\ &= -(\sigma - 1) \frac{1 - G(\varphi^*)}{\varphi^*} \left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*} \right)^{\sigma-1} < 0 \\ \frac{j'(\varphi^*)\varphi^*}{j(\varphi^*)} &= -(\sigma - 1) \left[1 + \frac{1}{\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*} \right)^{\sigma-1} - 1} \right] < -(\sigma - 1) \end{aligned}$$

Since $j(\varphi^*)$ is nonnegative and its elasticity is negative and bounded away from zero, $j(\varphi^*)$ is decreasing to zero as φ^* goes to infinity. Furthermore, $\lim_{\varphi^* \rightarrow 0} j(\varphi^*) = \infty$. Therefore $j(\varphi^*)$ decreases from infinity to zero on $(0, \infty)$ and the equilibrium cutoff exists and is unique.

2.7 Selection of firms in open economy

In open economy, two productivity cutoffs have to be determined, one for domestic production and one for exports:

$$\varphi^* = \inf\{\varphi : \pi(\varphi) > 0\}$$

$$\varphi_x^* = \inf\{\varphi : \pi_x(\varphi) > 0\}$$

Two cases to consider:

$$1) \varphi^* = \varphi_x^* \text{ and the cutoff firm is such that } \pi(\varphi^*) = \pi_d(\varphi^*) + \pi_x(\varphi^*) = 0$$

$$2) \varphi^* < \varphi_x^* \text{ and the cutoff firms are such that } \pi_d(\varphi^*) = 0 \text{ and } \pi_x(\varphi_x^*) = 0$$

Consider the second case:

$$\pi_d(\varphi^*) = 0 \Leftrightarrow p_d(\varphi^*)q_d(\varphi^*) = \sigma f$$

and:

$$\pi_x(\varphi_x^*) = 0 \Leftrightarrow p_x(\varphi_x^*)q_x(\varphi_x^*) = \sigma f_x \Leftrightarrow \tau^{1-\sigma} p_d(\varphi_x^*)q_d(\varphi_x^*) = \sigma f_x$$

The condition $\varphi^* < \varphi_x^*$ is thus satisfied iif:

$$p_d(\varphi^*)q_d(\varphi^*) < p_d(\varphi_x^*)q_d(\varphi_x^*) \Leftrightarrow \tau^{\sigma-1}f_x > f$$

The trade costs relative to the overhead production cost must be large enough. If the fixed cost of exporting is low enough, no firm find it profitable to sell in the domestic market while not exporting.

2.8 Aggregation in open economy

Ex-ante probability of successful entry:

$$p_{in} = 1 - G(\varphi^*)$$

Ex-ante probability that one of these successful firms will export:

$$p_x = \frac{1 - G(\varphi_x^*)}{1 - G(\varphi^*)}$$

If M is the equilibrium mass of incumbents in any country, $M_x = p_x M$ is the mass of exporting firms while $M_t = M + M_x$ is the total mass of varieties available for consumption.

The weighted productivity average thus becomes:

$$\tilde{\varphi}_t(\varphi^*, \varphi_x^*) = \left\{ \frac{1}{M_t} \left[M \tilde{\varphi}(\varphi^*)^{\sigma-1} + M_x \left(\frac{\tilde{\varphi}_x(\varphi_x^*)}{\tau} \right)^{\sigma-1} \right] \right\}^{\frac{1}{\sigma-1}}$$

where:

$$\begin{aligned} \tilde{\varphi}(\varphi^*) &= \left[\frac{1}{1 - G(\varphi^*)} \int_{\varphi^*}^{\infty} \varphi^{\sigma-1} g(\varphi) d\varphi \right]^{\frac{1}{\sigma-1}} \\ \tilde{\varphi}_x(\varphi_x^*) &= \left[\frac{1}{1 - G(\varphi_x^*)} \int_{\varphi_x^*}^{\infty} \varphi^{\sigma-1} g(\varphi) d\varphi \right]^{\frac{1}{\sigma-1}} \end{aligned}$$

The aggregate price thus becomes:

$$\begin{aligned}
P &= \left[\int_{\omega \in \Omega} p(\omega)^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \\
&= \left[Mp_d(\tilde{\varphi}(\varphi^*))^{1-\sigma} + M_x (\tau p_d(\tilde{\varphi}_x(\varphi_x^*)))^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \\
&= \frac{\sigma}{\sigma-1} w \left[M \tilde{\varphi}(\varphi^*)^{\sigma-1} + M_x \left(\frac{\tilde{\varphi}_x(\varphi_x^*)}{\tau} \right)^{\sigma-1} \right]^{\frac{1}{1-\sigma}} \\
&= M_t^{\frac{1}{1-\sigma}} \frac{\sigma}{\sigma-1} \frac{w}{\tilde{\varphi}_t}
\end{aligned}$$

Together with M_t , $\tilde{\varphi}_t$ entirely determines aggregate variables:

$$\begin{aligned}
R &= \int_{\omega \in \Omega} p(\omega)q(\omega)d\omega \\
&= Mp_d(\tilde{\varphi})q_d(\tilde{\varphi}) + M_x p_x(\tilde{\varphi}_x)q_x(\tilde{\varphi}_x) \\
&= R \left(\frac{\sigma}{\sigma-1} P \right)^{\sigma-1} \left[M \tilde{\varphi}^{\sigma-1} + M_x \left(\frac{\tilde{\varphi}_x}{\tau} \right)^{\sigma-1} \right] \\
&= R \left(\frac{\sigma}{\sigma-1} P \tilde{\varphi}_t \right)^{\sigma-1} M_t \\
Q &= \frac{R}{P} \\
&= M_t^{\frac{\sigma}{\sigma-1}} q_d(\tilde{\varphi}_t)
\end{aligned}$$

2.9 Zero cutoff profit condition in open economy

Rewrite the productivity cutoff conditions:

$$\begin{aligned}
\pi_d(\varphi^*) = 0 &\Leftrightarrow p_d(\varphi^*)q_d(\varphi^*) = \sigma f \\
&\Leftrightarrow p_d(\tilde{\varphi})q_d(\tilde{\varphi}) \left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*} \right)^{1-\sigma} = \sigma f \\
&\Leftrightarrow \pi_d(\tilde{\varphi}) = f \left[\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*} \right)^{\sigma-1} - 1 \right] \\
\pi_x(\varphi_x^*) = 0 &\Leftrightarrow p_x(\varphi_x^*)q_x(\varphi_x^*) = \sigma f_x \\
&\Leftrightarrow p_x(\tilde{\varphi}_x)q_x(\tilde{\varphi}_x) \left(\frac{\tilde{\varphi}_x(\varphi_x^*)}{\varphi_x^*} \right)^{1-\sigma} = \sigma f_x \\
&\Leftrightarrow \pi_x(\tilde{\varphi}_x) = f \left[\left(\frac{\tilde{\varphi}_x(\varphi_x^*)}{\varphi_x^*} \right)^{\sigma-1} - 1 \right]
\end{aligned}$$

Moreover:

$$\begin{aligned}
&\pi_d(\varphi^*) = 0 \quad \text{and} \quad \pi_x(\varphi_x^*) = 0 \\
\Rightarrow \frac{f_x}{f} &= \frac{p_x(\varphi_x^*)q_x(\varphi_x^*)}{p_d(\varphi^*)q_d(\varphi^*)} = \tau^{1-\sigma} \left(\frac{\varphi_x^*}{\varphi^*} \right)^{\sigma-1} \\
\Leftrightarrow \varphi_x^* &= \varphi^* \tau \left(\frac{f_x}{f} \right)^{\frac{1}{\sigma-1}}
\end{aligned}$$

Finally,

$$\bar{\pi} = \pi_d(\tilde{\varphi}) + p_x \pi_x(\tilde{\varphi}_x)$$

Combined together, these equations implicitly define $\bar{\pi}$ as a function of φ^* .

2.10 Equilibrium in open economy